Problem Set 1, Solutions
Math Camp 2025, UCSB
Instructors: Camilo Abbate and Sofía Olguín

1. Consider the following matrices:

$$\mathcal{A} = \begin{bmatrix} 2 & 0 \\ 3 & 8 \end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix} 7 & 2 \\ 6 & 3 \end{bmatrix}$$

   (a) Check $(\mathcal{A}\mathcal{B})^T = \mathcal{B}^T\mathcal{A}^T$.

$$(\mathcal{A}\mathcal{B})^T = \begin{bmatrix} 14 & 69 \\ 4 & 30 \end{bmatrix}; \quad \mathcal{B}^T\mathcal{A}^T = \begin{bmatrix} 7 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 14 & 69 \\ 4 & 30 \end{bmatrix}$$

   (b) Check $(\mathcal{A}\mathcal{B})^{-1} = \mathcal{B}^{-1}\mathcal{A}^{-1}$.

$$(\mathcal{A}\mathcal{B})^{-1} = \frac{1}{144}\begin{bmatrix} 30 & -4 \\ -69 & 14 \end{bmatrix};$$
$$\mathcal{B}^{-1}\mathcal{A}^{-1} = \frac{1}{9}\cdot\frac{1}{16}\begin{bmatrix} 3 & -2 \\ -6 & 7 \end{bmatrix}\begin{bmatrix} 8 & 0 \\ -3 & 2 \end{bmatrix} = \frac{1}{144}\begin{bmatrix} 30 & -4 \\ -69 & 14 \end{bmatrix}$$

   (c) Check $(\mathcal{A}^T)^{-1} = (\mathcal{A}^{-1})^T$.

$$(\mathcal{A}^T)^{-1} = \begin{bmatrix} 2 & 3 \\ 0 & 8 \end{bmatrix}^{-1} = \frac{1}{16}\begin{bmatrix} 8 & -3 \\ 0 & 2 \end{bmatrix} \quad (\mathcal{A}^{-1})^T = \frac{1}{16}\begin{bmatrix} 8 & -3 \\ 0 & 2 \end{bmatrix}$$

2. Let A and B be $n \times n$ matrices, where n is a positive integer ($n \geq 1$). Prove whether the following is true or false:

$$\det(A+B) = \det(A) + \det(B)$$

False.
Proof. Consider the following two matrices

$$\mathcal{A} = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix},$$

where k is a non-zero integer. Then $\det(A) = 0$ and $\det(B) = 0$, so $\det(A) + \det(B) = 0$. However, $\det(A+B) = \det(k \cdot \mathcal{I}_2) = k^2$, which is non-zero. ■

3. For each function below, determine whether it is convex or concave using its Hessian matrix.

   (a) $f(x,y) = -x^2 - y^2 + xy + 2x - y$

   (b) $g(a,b,c) = 3a^2 + 2b^2 + c^2 - 2ab + 2bc - 6a - 4b - 2c$

   (a) Let us first obtain the partials:

$$\frac{\partial f(x,y)}{\partial x} := f_x = -2x + y + 2$$
$$\frac{\partial f(x,y)}{\partial y} := f_y = -2y + x - 1$$

   Then the Hessian matrix of function $f(x,y)$ is a 2×2 matrix given by:

$$H_{f(x,y)} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

   We observe that $|H_1| < 0$ and $|H_2| > 0$. Since all of the leading principal minors of the Hessian matrix alternate the sign starting with the negative value, the matrix is negative definite by Theorem 5.10. Thus, by Theorem 5.12, $f(x,y)$ is strictly concave.

   (b) We start again with the partials:

$$\frac{\partial g(a,b,c)}{\partial a} := g_a = 6a - 2b - 6$$
$$\frac{\partial g(a,b,c)}{\partial b} := g_b = 4b - 2a + 2c - 4$$
$$\frac{\partial g(a,b,c)}{\partial c} := g_c = 2c + 2b - 2$$

   Then the Hessian matrix of function $g(a,b,c)$ is a 3×3 matrix given by:

$$H_{g(a,b,c)} = \begin{bmatrix} g_{aa} & g_{ab} & g_{ac} \\ g_{ba} & g_{bb} & g_{bc} \\ g_{ca} & g_{cb} & g_{cc} \end{bmatrix} = \begin{bmatrix} 6 & -2 & 0 \\ -2 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

   The leading principal minors are $|H_1| = 6$, $|H_2| = 20$, and $|H_3| = 16$. Thus, the matrix is positive definite by Theorem 5.10, and $g(a,b,c)$ is strictly convex by Theorem 5.12.

4. Let X be a $n \times k$ matrix with $\text{rank}(\mathbf{X}) = k (n \geq k)$. An annihilator matrix $\mathcal{M}_{\mathbf{X}}$ is

$$\mathcal{M}_{\mathbf{X}} = \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T.$$

   Show that $\mathcal{M}_{\mathbf{X}}$ is symmetric and idempotent.

$\mathcal{M}_{\mathbf{X}}$ is symmetric:

$$\begin{aligned}
\mathcal{M}_{\mathbf{X}}^T &= (\mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)^T \\
&= \mathcal{I}_n - (\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)^T & \mathcal{I}_n^T = \mathcal{I}_n \\
&= \mathcal{I}_n - \mathbf{X}((\mathbf{X}^T\mathbf{X})^{-1})^T\mathbf{X}^T & (\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \\
&= \mathcal{I}_n - \mathbf{X}((\mathbf{X}^T\mathbf{X})^T)^{-1}\mathbf{X}^T & (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T \\
&= \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T & (\mathbf{A}^T)^T = \mathbf{A} \\
&= \mathcal{M}_{\mathbf{X}}.
\end{aligned}$$

$\mathcal{M}_{\mathbf{X}}$ is idempotent:

$$\begin{aligned}
\mathcal{M}_{\mathbf{X}}\mathcal{M}_{\mathbf{X}} &= (\mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)(\mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) \\
&= \mathcal{I}_n - 2(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) + \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T \\
&= \mathcal{I}_n - 2(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) + \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T & \text{(cancel out } \mathbf{X}^T\mathbf{X}\text{)} \\
&= \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T \\
&= \mathcal{M}_{\mathbf{X}}.
\end{aligned}$$

5. Find eigenvalues and eigenvectors of the following matrix. Normalize the norm to one.

$$\mathcal{C} = \begin{bmatrix} .8 & .05 \\ .2 & .95 \end{bmatrix}$$

$$|(\mathcal{C} - \lambda I_2)| = \begin{vmatrix} \frac{4}{5} - \lambda & \frac{1}{20} \\ \frac{1}{5} & \frac{19}{20} - \lambda \end{vmatrix} = \left(\frac{4}{5} - \lambda\right)\left(\frac{19}{20} - \lambda\right) - \frac{1}{100} = \lambda^2 - \frac{7}{4}\lambda + \frac{3}{4} = (\lambda - 1)\left(\lambda - \frac{3}{4}\right).$$

Obtaining eigenvectors corresponding to each eigenvalue yields

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{17}} \\ \frac{4}{\sqrt{17}} \end{bmatrix} \quad \text{and} \quad \lambda_2 = \frac{3}{4}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

6. Express $-\sum_{i=1}^n \frac{u_i^2}{2\sigma^2}$ and $\sum_{i=1}^n \lambda_i u_i^2$ into quadratic forms using

$$\mathbf{u} := [u_1 \ \cdots \ u_n]', \quad \Sigma := \sigma^2\mathcal{I}_n, \quad \text{and} \quad \Lambda := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$-\sum_{i=1}^{n}\frac{u_i^2}{2\sigma^2} = -\frac{1}{2}\mathbf{u}^T\Sigma^{-1}\mathbf{u} \quad \text{and} \quad \sum_{i=1}^{n}\lambda_i v_i^2 = \mathbf{u}^T\Lambda\mathbf{u}$$

7. Consider the following regression equation where $n > k$:

$$\begin{aligned} y_i &= x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{ik-1}\beta_{k-1} + \beta_k + u_i && \forall\ i = 1, \cdots, n \\ &= \mathbf{x}_i{}^T\boldsymbol{\beta} + u_i && \forall\ i = 1, \cdots, n \end{aligned}$$

We can rewrite this as matrix equation:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{1k-1} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nk-1} & 1 \end{bmatrix} \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

(a) Check the following: $\mathbf{X}^T\mathbf{X} = \sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T$ and $\mathbf{X}^T\mathbf{y} = \sum_{i=1}^{n}\mathbf{x}_i y_i$.

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1k-1} & \cdots & x_{nk-1} \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1k-1} & 1 \\ \vdots & & \vdots & \vdots \\ x_{n1} & \cdots & x_{nk-1} & 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_{i1}^2 & \sum x_{i1}x_{i2} & \cdots & \sum x_{i1} \\ \vdots & \vdots & \vdots & \vdots \\ \sum x_{i1} & \sum x_{i2} & \cdots & n \end{bmatrix}$$

$$\mathbf{x_i}\mathbf{x_i}^T = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{ik-1} \\ 1 \end{bmatrix} \begin{bmatrix} x_{i1} & \cdots & x_{ik-1} & 1 \end{bmatrix} = \begin{bmatrix} x_{i1}^2 & x_{i1}x_{i2} & \cdots & x_{i1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{i1} & x_{i2} & \cdots & 1 \end{bmatrix}$$

$$\mathbf{X}^T\mathbf{y} = \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & & \vdots \\ x_{1k-1} & \cdots & x_{nk-1} \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum x_{i1}y_i \\ \sum x_{i2}y_i \\ \vdots \\ \sum y_i \end{bmatrix} = \sum_{i=1}^{n}\mathbf{x}_i y_i.$$

(b) Assume that $\operatorname{rank}(\mathbf{X}) = k$. We derived $\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$ in class. Show that

$$(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{y}_i\right)$$

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \left(\frac{1}{n}\mathbf{X}^T\mathbf{X}\right)^{-1}\left(\frac{1}{n}\mathbf{X}^T\mathbf{y}\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{y}_i\right)$$

(c) Show that

$$\sqrt{n}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right).$$

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{y}_i\right) = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i(\mathbf{x}_i^T\boldsymbol{\beta} + \mathbf{u}_i)\right) \\
&= \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)\boldsymbol{\beta} + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right) \\
&= \boldsymbol{\beta} + \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i\mathbf{u}_i\right)
\end{aligned}$$

(d) Let $\hat{\mathbf{u}} := \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}}$. Show that $\sum_{i=1}^{n}\hat{u}_i^2 = \mathbf{u}^T\mathcal{M}_{\mathbf{X}}\mathbf{u}$ where $\mathcal{M}_{\mathbf{X}} = \mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$.

$$\begin{aligned}
\hat{\mathbf{u}} &= \mathbf{y} - \mathbf{X}\widehat{\boldsymbol{\beta}} \\
&= \mathbf{y} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} \\
&= \left(\mathcal{I}_n - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\mathbf{y} \\
&= \mathcal{M}_{\mathbf{X}}(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\
&= \mathcal{M}_{\mathbf{X}}\mathbf{u}. && (\mathcal{M}_{\mathbf{X}}\mathbf{X} = \mathbf{0}_n)
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{u}}^T\hat{\mathbf{u}} &= \left(\mathcal{M}_{\mathbf{X}}\mathbf{u}\right)^T\left(\mathcal{M}_{\mathbf{X}}\mathbf{u}\right) \\
&= \mathbf{u}^T\mathcal{M}_{\mathbf{X}}^T\mathcal{M}_{\mathbf{X}}\mathbf{u} \\
&= \mathbf{u}^T\mathcal{M}_{\mathbf{X}}^2\mathbf{u} && \text{(Symmetric)} \\
&= \mathbf{u}^T\mathcal{M}_{\mathbf{X}}\mathbf{u}. && \text{(Idempotent)}
\end{aligned}$$