Problem Set 1 - Analysis - Solutions[1]

Question 1 Suppose P, Q, and R are statements. Use the truth table to show that the following statements are always true.

(1) $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ (*modus ponens*)

(2) $((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$ (*modus tollens*)

(3) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ (*syllogism*)

(1) *Modus ponens*

P	Q	$P \Rightarrow Q$	$(P \wedge (P \Rightarrow Q))$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(2) *Modus tollens*

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$((P \Rightarrow Q) \wedge \neg Q)$	$((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

(3) *Syllogism*

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R))$	$(P \Rightarrow R)$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

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Question 2 Suppose A and B are sets. Show the following statements are equivalent.

(1) $A \subset B$.

(2) $A \cup B = B$.

(3) $A \cap B = A$.

Let us first show (1) $\iff$ (2):

- (1) $\Rightarrow$ (2)
  (i) We show that $A \cup B \subset B$: Let us take $x \in A \cup B$. By definition, either $x \in A$ or $x \in B$ (or both). If $x \in A$, then $x \in B$ because $A \subset B$. Thus, in either case, $x \in B$.
  (ii) We show that $B \subset A \cup B$: Let us take $x \in B$. By definition of union, $x \in A \cup B$.
  Therefore, we showed that if $A \subset B$ then $A \cup B = B$.
- (2) $\Rightarrow$ (1)
  Let us take $x \in A$. By definition of union $x \in A \cup B$. Since $A \cup B = B$, then $x \in B$.
  Therefore, $A \subset B$.

$\therefore A \subset B \iff A \cup B = B$

Next, let us show (1) $\iff$ (3):

- (1) $\Rightarrow$ (3)
  (i) We show that $A \cap B \subset A$: Let us take $x \in A \cap B$. By definition of intersection, $x \in A$ and $x \in B$.
  (ii) We show that $A \subset A \cap B$: Let us take $x \in A$. Since $A \subset B$, then $x \in B$. Thus, by definition of intersection, $x \in A \cap B$.
  Therefore, we showed that if $A \subset B$ then $A \cap B = A$.
- (3) $\Rightarrow$ (1)
  Let us take $x \in A$. Since $A = A \cap B$ and by definition of intersection, $x \in B$. Therefore, $A \subset B$.

$\therefore A \subset B \iff A \cap B = A$

Finally, since statement (1) is equivalent to both statements (2) and (3), then (2) $\iff$ (3).

Question 3 In class, we defined the uniqueness existential quantifier: $\exists!$. For example, "$\exists! x \in X$ such that..." means "there exists a unique x in X such that..." However, such statements can be defined using $\forall$, $\exists$, and logical operators.
Write a symbolic statement that is equivalent to "$\exists! x \in X$ such that P(X)" without using !.

$$\exists x \in X \text{ such that } P(x) \wedge \forall y \in X \text{ such that } P(y) \Rightarrow y = x$$

Question 4 Let us consider a function $f : \mathbb{R}^n \to \mathbb{R}$.

(a) If f is strictly increasing, is it strongly increasing?

(b) If f is strongly increasing, is it strictly increasing?

(a) If f is strictly increasing does not imply that it is strongly increasing. Counterexample: Let us consider $f(x_1, x_2) = \min\{x_1, x_2\}$. We showed in the lecture notes that f is strictly increasing. However, it is not strongly increasing. To see this, let us consider $\mathbf{x} = (2, 6)$ and $\mathbf{y} = (2, 5)$. Notice that $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} \geq \mathbf{y}$. However, $f(\mathbf{x}) = f(\mathbf{y}) = 2$. Therefore, f is not strongly increasing.

(b) WTS: if f is strongly increasing then it is strictly increasing.
Let us consider $\mathbf{x} \gg \mathbf{y}$, that is, $x_i > y_i$ for every $i = 1, ..., n$. This implies that $\mathbf{x} \neq \mathbf{y}$. Since f is strongly increasing, then $f(\mathbf{x}) > f(\mathbf{y})$. Therefore, f is strictly increasing.

Question 5 Use the epsilon-delta definition of a limit to prove that

$$\lim_{x \to 1} \frac{x}{x^2 + 1} = \frac{1}{2}$$

In this example, $L = \frac{1}{2}$, $a = 1$, and $f(x) = \frac{x}{x^2+1}$.

Step 1: Choose an open interval containing a.
Let us consider $x \in (0, 2)$.

Step 2: Start with the inequality $|f(x) - L| < \varepsilon$.

$$|f(x) - L| = \left| \frac{x}{x^2 + 1} - \frac{1}{2} \right| = \left| \frac{-(x-1)^2}{2(x^2 + 1)} \right| < \frac{(x-1)^2}{2}$$

The inequality holds because we consider $x \in (0, 2)$.

For an arbitrary $\varepsilon > 0$, the inequality $|f(x) - L| < \varepsilon$ holds if:

$$\frac{(x-1)^2}{2} < \varepsilon \quad \text{and} \quad x \in (0, 2)$$

$$|x - 1| < \sqrt{2\varepsilon} \quad \text{and} \quad |x - 1| < 1$$

Step 3: Choose $\delta > 0$ to make the inequality hold.

When we are given an arbitrary $\varepsilon > 0$, no matter how small ε is, we can always choose a $\delta = \min\{1, \sqrt{2\varepsilon}\}$ accordingly to have:

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Question 6 Use the Squeeze Theorem, prove that

$$\lim_{x\to\infty}(1 + \frac{1}{x^2})^x = 1$$

(Hint: $\lim_{x\to\infty}(1 + \frac{1}{x})^x = e$ and $\lim_{x\to\infty} e^{\frac{1}{x}} = 1$)

Let $g(x) = (1+\frac{1}{x^2})^x$. We need to find functions $f(x)$ and $h(x)$ such that $f(x) \leq g(x) \leq h(x)$.

Let $f(x) = 1$. Since $\frac{1}{x^2} > 0 \Rightarrow 1 + \frac{1}{x^2} > 1$ then so is $(1 + \frac{1}{x^2})^x$. Therefore, $f(x) \leq g(x)$ and $\lim_{x\to\infty} f(x) = 1$.

Let $h(x) = e^{1/x}$. Since $\ln\left(1 + \frac{1}{x^2}\right) \leq \frac{1}{x^2} \Rightarrow x\ln\left(1 + \frac{1}{x^2}\right) \leq \frac{1}{x} \Rightarrow \left(1 + \frac{1}{x^2}\right)^x \leq e^{1/x}$ for $x > 0$. And from the hint $\lim_{x\to\infty} h(x) = 1$

Thus, we have:

$$f(x) \leq g(x) \leq h(x)$$

$$1 \leq \left(1 + \frac{1}{x^2}\right)^x \leq e^{1/x}$$

and both $f(x)$ and $h(x)$ converge to 1 as $x \to \infty$.

Therefore, $\lim_{x\to\infty}\left(1 + \frac{1}{x^2}\right)^x = 1$.