Problem Set 2 - Analysis - Solutions

Question 1 Let x and y be any real numbers. Show that $||x| - |y|| \leq |x - y|$ (this is called the reverse triangle inequality)

Proof. Note that for any two real numbers a and b, we can always write $|a| + |b| \geq |a + b|$. Therefore we can write:

$|x| + |y - x| \geq |x + y - x| = |y|$ and $|y| + |x - y| \geq |y + x - y| = |x|$

Rewritting these two inequalities, we have:

$|y - x| \geq |y| - |x|$ and $|x - y| \geq |x| - |y|$

From the absolute value properties, we know that $|y - x| = |x - y|$, and if $t \geq a$ and $t \geq -a$, then $t \geq |a|$.

Combining these facts together:

$|x - y| \geq ||x| - |y||$

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Question 2 Find the Taylor polynomial of degree 5 ($n = 5$) for the function $f(x) = e^x$ around the point $x = 0$. Then, evaluate the polynomial at $x = 1$.

The Taylor polynomial of degree 5 for $f(x) = e^x$ centered at $x = 0$ is:

$$P_5(x) = \sum_{k=0}^{5} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{5} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.$$

Evaluating at $x = 1$:

$$P_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{326}{120} = \frac{163}{60} \approx 2.71667.$$

Thus,

$$\boxed{P_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}} \quad \text{and} \quad \boxed{P_5(1) = \frac{163}{60} \approx 2.71667}.$$

Question 3 Let $x(t)$ be differentiable. Show that $\frac{\frac{dx(t)}{dt}}{x} = \frac{d\log(x(t))}{dt}$.

Consider that $g()$ is the natural logarithm function, i.e., $g(x) = \log(x)$. Then, we can express $\log(x(t))$ as a composition of functions: $g(x(t))$. Applying the Chain Rule:

$$\frac{d\log(x(t))}{dt} = \frac{d\log(x(t))}{dx(t)} \cdot \frac{dx(t)}{dt} = \frac{1}{x(t)} \cdot \frac{dx(t)}{dt}$$

Question 4 Determine whether the following functions are convex/concave/quasi-convex/quasi-concave.
(1) $f(x, y) = \sqrt{x} + \sqrt{y}$
(2) $g(x, y) = \sqrt{xy}$

(1)

If we construct the Hessian matrix of $f(x, y)$, we have:

$$H = \begin{bmatrix} -\frac{1}{4x^{3/2}} & 0 \\ 0 & -\frac{1}{4y^{3/2}} \end{bmatrix}$$

The leading principal minors of the Hessian are:

$$D_1 = -\frac{1}{4x^{3/2}} < 0$$

$$D_2 = \left(-\frac{1}{4x^{3/2}}\right)\left(-\frac{1}{4y^{3/2}}\right) - 0 = \frac{1}{16x^{3/2}y^{3/2}} > 0$$

Since the leading principal minors alternate in sign, the Hessian is negative definite. Therefore, $f(x, y)$ is strictly concave.

(2)

If we construct the Hessian matrix of $g(x, y)$, we have:

$$H = \begin{bmatrix} -\frac{1}{4}x^{-3/2}y^{1/2} & \frac{1}{4}x^{-1/2}y^{-1/2} \\ \frac{1}{4}x^{-1/2}y^{-1/2} & -\frac{1}{4}x^{1/2}y^{-3/2} \end{bmatrix}$$

The leading principal minors of the Hessian are:

$$D_1 = -\frac{1}{4}x^{-3/2}y^{1/2} < 0$$

$$D_2 = \left(-\frac{1}{4}x^{-3/2}y^{1/2}\right)\left(-\frac{1}{4}x^{1/2}y^{-3/2}\right) - \left(\frac{1}{4}x^{-1/2}y^{-1/2}\right)^2 = 0$$

Since the second leading principal minor is zero, the Hessian is negative semidefinite. Therefore, $g(x, y)$ is concave.

Question 5

Let (X, d) be a metric space, where $X = \mathbb{R}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}$, define $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.

(1) Consider $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right), n = 1, 2, \ldots.$ For any n, is I_n an open set or a closed set (or both)? Find $\bigcap_{n=1}^{\infty} I_n$. Is it an open set or a closed set (or both)?

(2) Consider $J_n = \left[\dfrac{1}{n+1}, 1 - \dfrac{1}{n+1}\right], n = 1, 2, ...$ For any n, is J_n an open set or a closed set (or both)? Find $\bigcup_{n=1}^{\infty} J_n$. Is it an open set or a closed set (or both)?

(1)

First, we are going to prove that any open interval (c, d) is an open set in $\mathbb{R}$.

Let $c < d$ be two real numbers.

Hence, $(c, d) = \{x \in \mathbb{R} \mid c < x < d\}$.

Let a be any point in (c, d). That is, $c < a < d$. We need to show that there exists an $\epsilon > 0$ such that the open ball (or neighborhood) $V_\epsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \epsilon\}$ is contained within the interval (c, d).

Choose $\epsilon < \min\{a - c, d - a\}$.

Picture it geometrically by drawing a real line. $|x - a| < \epsilon$ represents all points on the line that are ϵ-distant from the point a. By picking $\epsilon < \min\{a - c, d - a\}$ what you do is to pick the smallest distance from the point a to the boundaries of the interval. Now if we create a neighbourhood (an open set) around a again using this minimum distance it will clearly be contained in the original interval.

Rigorously,

$$x \in V_\epsilon(a) \implies |x - a| < \epsilon \iff a - \epsilon < x < a + \epsilon$$

(1)

Now $\epsilon \leq a - c$ and $\epsilon \leq d - a$. Use these to approximate ϵ in (1). That is,

$$c = a - (a - c) < a - \epsilon < x < a + \epsilon < a + (d - a) = d \iff x \in (c, d) \implies V_\epsilon(a) \subseteq (c, d)$$

Now that we have proven this, let's show that $\bigcap_{n=1}^{\infty} I_n$ is a closed set.

For each n, $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ is an open interval, so it is an **open set** in $\mathbb{R}$.

Now, consider the intersection:

$$\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right).$$

For any $x \neq 0$, there exists N such that $|x| > \frac{1}{N}$, so $x \notin I_n$ for $n \geq N$. Only $x = 0$ belongs to every I_n. Therefore,

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$

Notice that any sequence $\{x_k\}$ in $\{0\}$ converges to 0, which is in the set. Thus, $\{0\}$ contains all its limit points and is a **closed set**.

(2)

Let's start by proving that any closed interval $[c, d]$ is a closed set in $\mathbb{R}$.

We can take advantage of what we have already proven in (1), i.e. any open interval is an open set. Now, we also know from the lecture notes that the union of two open sets is also an open set.

Recall that the complement of a closed set is open. The complement of $[c, d]$ in $\mathbb{R}$ is:

$$\mathbb{R} \setminus [c, d] = (-\infty, c) \cup (d, \infty).$$

Both $(-\infty, c)$ and (d, ∞) are open intervals, hence open sets. Their union is also open. Therefore, the complement of $[c, d]$ is open, which means $[c, d]$ is closed.

Using this result, we can state then that for any N, J_n is a closed set.

Now, let's consider $\bigcup\limits_{n=1}^{\infty} J_n$.

$$\bigcup_{n=1}^{\infty} J_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, 1 - \frac{1}{n+1}\right]$$

Since $\frac{1}{n+1}$ approaches 0 as n approaches infinity, and $1 - \frac{1}{n+1}$ approaches 1, the union of all these intervals is the open interval $(0, 1)$.

Since all open intervals are open sets, we conclude that $(0, 1)$ is an **open set**.

Question 6

(Comment: You can use Euler's theorem for homogeneous functions here, but only after proving it.)

(1) Let $Q = f(K, L)$ be a differentiable homogeneous function of degree one. Prove that

$$f(K, L) = K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L}$$

(2) Let $Q = f(K, L)$ be a differentiable homogeneous function of degree r. We can also have a similar formula:

$$f(K, L) = g(r)(K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L})$$

where $g(r)$ is some function of r. Find $g(r)$.

(1)

Let $\lambda \in \mathbb{R}_+$

$$f(\lambda K, \lambda L) = \lambda f(K, L)$$

Because f is homogeneous of degree 1, we can differentiate both sides with respect to λ:

$$\frac{\partial f(\lambda K, \lambda L)}{\partial K}\frac{\partial(\lambda K)}{\partial \lambda} + \frac{\partial f(\lambda K, \lambda L)}{\partial L}\frac{\partial(\lambda L)}{\partial \lambda} = f(K, L)$$

In particular, if $\lambda = 1$, we have

$$f(K, L) = \frac{\partial f(K, L)}{\partial K}K + \frac{\partial f(K, L)}{\partial L}L = K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L}$$

(2)

Again, let $\lambda \in \mathbb{R}_+$. Now, because f is homogeneous of degree r, we have

$$f(\lambda K, \lambda L) = \lambda^r f(K, L)$$

Differentiating both sides with respect to λ:

$$\frac{\partial f(\lambda K, \lambda L)}{\partial \lambda} = \frac{\partial f(\lambda K, \lambda L)}{\partial K}\frac{\partial (\lambda K)}{\partial \lambda} + \frac{\partial f(\lambda K, \lambda L)}{\partial L}\frac{\partial (\lambda L)}{\partial \lambda} = \frac{d}{d\lambda}\left(\lambda^r f(K, L)\right) = r\lambda^{r-1} f(K, L)$$

Once again, let $\lambda = 1$:

$$rf(K, L) = \frac{\partial f(K, L)}{\partial K}K + \frac{\partial f(K, L)}{\partial L}L = K\frac{\partial f(K, L)}{\partial K} + L\frac{\partial f(K, L)}{\partial L}$$

$$\Rightarrow f(K, L) = \frac{1}{r}\left(K\frac{\partial f(K, L)}{\partial K} + L\frac{\partial f(K, L)}{\partial L}\right)$$

meaning that $g(r) = \frac{1}{r}$.