Problem Set 3 - Analysis - Solutions[1]

Question 1 Let X be a random variable with $\mathbb{E}[X] = 0$. Show that $\mathbb{E}[X^2] > 0$ if X takes more than one value with positive probability.
(Hint: Use Jensen's inequality.)

Let X be a random variable with $\mathbb{E}[X] = 0$.
Let us define $g(x) = x^2$. Since $g(x)$ is a convex function, by Jensen's inequality:

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X])$$

$$\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$$

$$\mathbb{E}[X^2] \geq 0$$

If X takes more than one value with positive probability, then $\mathbb{E}[X^2] \neq 0$. Therefore,

$$\mathbb{E}[X^2] > 0$$

Question 2 Consider the following expression for the reservation wage w_R:

$$w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} (w - w_R)\, dF(w)$$

Use integration by parts to show that this expression can be rewritten as:

$$w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} [1 - F(w)]\, dw$$

where $F(w)$ is the (cumulative) distribution of wages, with support $(\underline{w}, \bar{w})$.
(Hint: consider how the CDF behaves at the boundaries of its support.)

Let us consider the integral in the expression:

$$w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\bar{w}} (w - w_R)\, dF(w)$$

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Let us define:

$$H(x) = (w - w_R) \quad h(x) = 1$$

$$g(x)dx = dF(w) \qquad G(x) = F(w) \quad g(x) = f(w)$$

Thus, applying integration by parts:

$$\int_{w_R}^{\bar{w}} (w - w_R)\, dF(w) = (\bar{w} - w_R)\, F(\bar{w}) - (w_R - w_R)\, F(w_R) - \int_{w_R}^{\bar{w}} F(w)dw$$

$$= (\bar{w} - w_R) - \int_{w_R}^{\bar{w}} F(w)dw$$

$$= \int_{w_R}^{\bar{w}} 1dw - \int_{w_R}^{\bar{w}} F(w)dw$$

$$= \int_{w_R}^{\bar{w}} [1 - F(w)]dw$$

Therefore,

$$w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} [1 - F(w)]\, dw$$

Question 3 Consider the following constrained utility maximization problem:

$$\max_{x_1, x_2} \alpha \ln(x_1) + \beta \ln(x_2) \quad \text{subject to} \quad m \geq p_1 x_1 + p_2 x_2, \quad x_1 \geq 0, \quad x_2 \geq 0$$

where $\alpha > 0$, $\beta > 0$, $m > 0$, $p_1 > 0$ and $p_2 > 0$.

(a) Find the demand functions (or correspondences) $x_1^*(p_1, p_2, w)$ and $x_2^*(p_1, p_2, w)$, assuming the budget constraint binds.

$$\max_{x_1 \geqslant 0,\ x_2 \geqslant 0} \alpha \ln(x_1) + \beta \ln(x_2) \qquad \text{s.t} \qquad p_1 x_1 + p_2 x_2 = m$$

Let us write the Lagrangian:

$$L(x_1, x_2, \lambda) = \alpha \ln(x_1) + \beta \ln(x_2) + \lambda [m - p_1 x_1 - p_2 x_2]$$

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$$\frac{\partial L}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0 \tag{1}$$

$$\frac{\partial L}{\partial x_2} = \frac{\beta}{x_2} - \lambda p_2 = 0 \tag{2}$$

$$\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0 \Rightarrow m = p_1 x_1 + p_2 x_2 \tag{3}$$

From (1) and (2):

$$\frac{\alpha}{\beta}\frac{x_2}{x_1} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = \frac{\beta}{\alpha}\frac{p_1}{p_2}x_1$$

Plugging into (3):

$$m = p_1 x_1 + p_2 \frac{\beta}{\alpha}\frac{p_1}{p_2}x_1 \quad \Rightarrow \quad x_1^* \left(p_1, p_2, m\right) = \left(\frac{\alpha}{\alpha+\beta}\right)\frac{m}{p_1}$$

Using this result in the expression for x_2: $\quad x_2^* \left(p_1, p_2, m\right) = \left(\frac{\beta}{\alpha+\beta}\right)\frac{m}{p_2}$

(b) Find the demand functions (or correspondences) using the Karush-Kuhn-Tucker conditions.

$$\max_{x_1 \geqslant 0,\ x_2 \geqslant 0} \alpha \ln\left(x_1\right) + \beta \ln\left(x_2\right) \qquad \text{s.t} \qquad p_1 x_1 + p_2 x_2 \leq m$$

Since $\ln(t)$ requires $x_1, x_2 > 0$ then the complementary slackness conditions would require $\mu_1 = 0$ and $\mu_2 = 0$. Thus, we only need to consider the budget constraint.
Let us write the Lagrangian:

$$L\left(x_1, x_2, \lambda\right) = \alpha \ln\left(x_1\right) + \beta \ln\left(x_2\right) + \lambda\left[m - p_1 x_1 - p_2 x_2\right]$$

KKT conditions:

$$\frac{\partial L}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\beta}{x_2} - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 \geqslant 0 \Rightarrow m \geqslant p_1 x_1 + p_2 x_2$$

$$\lambda \geqslant 0, \quad \text{and} \quad \lambda\left[m - p_1 x_1 - p_2 x_2\right] = 0$$

Case 1: $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$
Then $m = p_1 x_1 + p_2 x_2$, and it is the case solved in a).

Case 2: $\frac{\partial \mathcal{L}}{\partial \lambda} > 0$
Then $\lambda = 0 \Rightarrow$ From $\frac{\partial \mathcal{L}}{\partial x_1} : \frac{\alpha}{x_1} = 0$ which is not possible since $\alpha > 0$ and $x_1 > 0$.

Therefore, the only solution is the interior solution found in a).

(c) Verify that the demand functions solve the maximization problem. (You may refer to the theorems discussed in class.)

Since the utility function is differentiable and concave for $\mathbf{x} \geq 0$ and the budget constraint is convex, x_1^*, x_2^* solves the maximization problem, by the Kuhn-Tucker sufficiency theorem.

(d) Find the indirect utility function $v(p_1, p_2, m) = \alpha \ln(x_1^*) + \beta \ln(x_2^*)$.

$$\begin{aligned} v(p_1, p_2, m) &= \alpha \ln(x_1^*) + \beta \ln(x_2^*) \\ &= \alpha \ln\left(\left(\frac{\alpha}{\alpha+\beta}\right)\frac{m}{p_1}\right) + \beta \ln\left(\left(\frac{\beta}{\alpha+\beta}\right)\frac{m}{p_2}\right) \\ &= \alpha \ln\left(\frac{\alpha}{\alpha+\beta}\right) + \beta \ln\left(\frac{\beta}{\alpha+\beta}\right) + (\alpha+\beta)\ln(m) - \alpha\ln(p_1) - \beta\ln(p_2) \end{aligned}$$

Question 4 Consider the following constrained expenditure minimization problem:

$$\min_{h_1, h_2} \; p_1 h_1 + p_2 h_2 \quad \text{subject to} \quad \alpha \ln(h_1) + \beta \ln(h_2) \geq u, \quad h_1 \geq 0, \quad h_2 \geq 0$$

where $\alpha > 0$, $\beta > 0$, $u > 0$, $p_1 > 0$ and $p_2 > 0$.

(a) Find the demand functions (or correspondences) $h_1^*(p_1, p_2, u)$ and $h_2^*(p_1, p_2, u)$.

$$\min_{h_1 \geq 0,\; h_2 \geq 0} \; p_1 h_1 + p_2 h_2 \quad \text{subject to} \quad \alpha \ln(h_1) + \beta \ln(h_2) \geq u$$

Let us write the Lagrangian:

$$L(x_1, x_2, \mu) = -p_1 h_1 - p_2 h_2 + \mu \left[\alpha \ln(h_1) + \beta \ln(h_2) - u\right]$$

KKT conditions:

$$\frac{\partial L}{\partial h_1} = -p_1 + \alpha\mu\frac{1}{h_1} = 0 \qquad \Rightarrow \qquad p_1 = \alpha\mu\frac{1}{h_1} \quad (1)$$

$$\frac{\partial L}{\partial h_2} = -p_2 + \beta\mu\frac{1}{h_2} = 0 \qquad \Rightarrow \qquad p_2 = \beta\mu\frac{1}{h_2} \quad (2)$$

$$\frac{\partial L}{\partial \mu} = \alpha\ln(h_1) + \beta\ln(h_2) - u \geq 0$$

$$\mu \geq 0 \qquad \text{and} \qquad \mu\left[\alpha\ln(h_1) + \beta\ln(h_2) - u\right] = 0$$

From (1) and (2):

$$\frac{p_1}{p_2} = \frac{\alpha}{\beta}\frac{h_2}{h_1} \qquad \Rightarrow \qquad h_2 = \frac{\beta}{\alpha}\frac{p_1}{p_2}h_1 \quad (3)$$

<u>Case 1:</u> $\frac{\partial L}{\partial \mu} > 0$

In this case, the constraint is not binding, so the multiplier μ must be zero. Substituting $\mu = 0$ into equation (1) yields $p_1 = \alpha\mu\frac{1}{h_1} = 0$, which is not possible given the assumptions of the problem.

<u>Case 2:</u> $\frac{\partial L}{\partial \mu} = 0$

By this condition:

$$\alpha\ln(h_1) + \beta\ln(h_2) = u$$

Plugging in (3):

$$\alpha\ln(h_1) + \beta\ln\left(\frac{\beta}{\alpha}\frac{p_1}{p_2}h_1\right) = u$$

$$\alpha\ln(h_1) + \beta\ln\left(\frac{\beta}{\alpha}\frac{p_1}{p_2}\right) + \beta\ln\left(h_1\right) = u$$

$$(\alpha+\beta)\ln(h_1) = u - \beta\ln\left(\frac{\beta}{\alpha}\frac{p_1}{p_2}\right)$$

$$h_1^{\alpha+\beta} = e^u\left(\frac{\beta}{\alpha}\frac{p_1}{p_2}\right)^{-\beta}$$

$$h_1^*(p_1,p_2,u) = \exp\left\{\frac{u}{\alpha+\beta}\right\}\cdot\left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}}\cdot\left(\frac{p_1}{p_2}\right)^{-\frac{\beta}{\alpha+\beta}}$$

Plugging this result into (3):

$$
\begin{aligned}
h_2^*(p_1, p_2, u) &= \frac{\beta}{\alpha}\frac{p_1}{p_2} \exp\left\{\frac{u}{\alpha+\beta}\right\} \cdot \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} \cdot \left(\frac{p_1}{p_2}\right)^{-\frac{\beta}{\alpha+\beta}} \\
&= \exp\left\{\frac{u}{\alpha+\beta}\right\} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \cdot \left(\frac{p_1}{p_2}\right)^{\frac{\alpha}{\alpha+\beta}}
\end{aligned}
$$

(b) Find the expenditure function $e(p_1, p_2, u) = p_1 h_1^* + p_2 h_2^*$.

$e(p_1, p_2, u) = p_1 h_1^* + p_2 h_2^*$

$$
\begin{aligned}
&= p_1 \exp\left\{\frac{u}{\alpha+\beta}\right\} \cdot \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} \cdot \left(\frac{p_1}{p_2}\right)^{-\frac{\beta}{\alpha+\beta}} + p_2 \exp\left\{\frac{u}{\alpha+\beta}\right\} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \cdot \left(\frac{p_1}{p_2}\right)^{\frac{\alpha}{\alpha+\beta}} \\
&= \exp\left\{\frac{u}{\alpha+\beta}\right\} p_1^{\frac{\alpha}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} \cdot \left[\left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right]
\end{aligned}
$$

(c) Verify the duality property:

$$
x^*(p_1, p_2, e(p_1, p_2, u)) = h^*(p_1, p_2, u) \qquad \text{and} \qquad h^*(p_1, p_2, v(p_1, p_2, m)) = x^*(p_1, p_2, m).
$$

using the demands x_1^*, x_2^*, and indirect utility function v derived in Question 3.

First, consider the demand function $x_1^*(p_1, p_2, m)$, evaluated at $m = e(p_1, p_2, u)$:

$$
\begin{aligned}
x^*(p_1, p_2, e(p_1, p_2, u)) &= \left(\frac{\alpha}{\alpha+\beta}\right) \frac{e(p_1, p_2, u)}{p_1} \\
&= \left(\frac{\alpha}{\alpha+\beta}\right) \frac{1}{p_1} \exp\left\{\frac{u}{\alpha+\beta}\right\} p_1^{\frac{\alpha}{\alpha+\beta}} p_2^{\frac{\beta}{\alpha+\beta}} \cdot \left[\left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] \\
&= \exp\left\{\frac{u}{\alpha+\beta}\right\} \left(\frac{p_1}{p_2}\right)^{-\frac{\beta}{\alpha+\beta}} \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} \left(\frac{\alpha}{\alpha+\beta}\right) \left[1 + \frac{\beta}{\alpha}\right] \\
&= \exp\left\{\frac{u}{\alpha+\beta}\right\} \cdot \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} \cdot \left(\frac{p_1}{p_2}\right)^{-\frac{\beta}{\alpha+\beta}} \\
&= h_1^*(p_1, p_2, u)
\end{aligned}
$$

This verifies the first equality.

Similarly, evaluating the demand function $h_1^*(p_1, p_2, u)$ at $u = v(p_1, p_2, m)$:

$$\begin{aligned}
h_1^*(p_1, p_2, u) &= \exp\left\{\frac{u}{\alpha+\beta}\right\} \cdot \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} \cdot \left(\frac{p_1}{p_2}\right)^{-\frac{\beta}{\alpha+\beta}} \\
&= \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} \cdot \left(\frac{p_1}{p_2}\right)^{-\frac{\beta}{\alpha+\beta}} \exp\left\{\frac{\alpha}{\alpha+\beta} \ln\left(\left(\frac{\alpha}{\alpha+\beta}\right)\frac{m}{p_1}\right) + \frac{\beta}{\alpha+\beta} \ln\left(\left(\frac{\beta}{\alpha+\beta}\right)\frac{m}{p_2}\right)\right\} \\
&= \left(\frac{\beta}{\alpha}\right)^{-\frac{\beta}{\alpha+\beta}} \cdot \left(\frac{p_1}{p_2}\right)^{-\frac{\beta}{\alpha+\beta}} \left(\left(\frac{\alpha}{\alpha+\beta}\right)\frac{m}{p_1}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\left(\frac{\beta}{\alpha+\beta}\right)\frac{m}{p_2}\right)^{\frac{\beta}{\alpha+\beta}} \\
&= \left(\frac{\alpha}{\alpha+\beta}\right)\frac{m}{p_1} \\
&= x_1^*(p_1, p_2, m)
\end{aligned}$$

This confirms the second equality.

The duality property can be fully confirmed by applying the same procedure to the demand functions for good 2.

Question 5 Suppose the motion of capital, $\dot{k}$, satisfies the differential equation:

$$\dot{k} = 0.03k + 0.01$$

(a) Find the general solution ($k(t)$) to this ODE.
Applying the solution for Form 2:

$$\begin{aligned}
k(t) &= a\, e^{0.03t} - \frac{0.01}{0.03} \\
&= a\, e^{0.03t} - \frac{1}{3}
\end{aligned}$$

where a is a constant.

(b) Let $k(0) = 100$. Find the particular solution ($k(t)$) to this ODE.

$$k(0) = a - \frac{1}{3} \qquad \Rightarrow \qquad a = 100 + \frac{1}{3} \quad (\text{using } k(0) = 100)$$

Replacing the constant:

$$k(t) = \left(100 + \frac{1}{3}\right) e^{0.03t} - \frac{1}{3}$$

Question 6 Assume the capital-output ratio $x(t) = k(t)/y(t)$ evolves according to:

$$\dot{x}(t) = s_1(1-\alpha) - (\delta+n)(1-\alpha)x(t)$$

(a) Find the general solution $(x(t))$ to this ODE.

Let us rewrite the ODE:

$$\frac{dx(t)}{dt} + (\delta+n)(1-\alpha)x(t) = s_1(1-\alpha)$$

$$r(t)\frac{dx(t)}{dt} + r(t)(\delta+n)(1-\alpha)x(t) = r(t)s_1(1-\alpha) \qquad \text{(using an integrating factor } r(t))$$

$$\frac{d}{dt}x(t)r(t) = r(t)s_1(1-\alpha) \qquad \left(\text{with } \frac{dr(t)}{dt} = r(t)(\delta+n)(1-\alpha)\right)$$

$$x(t)r(t) = \int^t r(v)s_1(1-\alpha)dv \qquad \text{(integrating both sides)}$$

$$x(t)e^{(\delta+n)(1-\alpha)t} = \int^t e^{(\delta+n)(1-\alpha)v}s_1(1-\alpha)dv \qquad (\text{using } r(t) = e^{(\delta+n)(1-\alpha)t})$$

$$x(t) = e^{-(\delta+n)(1-\alpha)t}\left[\frac{s_1}{(\delta+n)}e^{(\delta+n)(1-\alpha)t} + a\right]$$

$$x(t) = a\ e^{-(\delta+n)(1-\alpha)t} + \frac{s_1}{(\delta+n)}$$

where a is a constant.

(b) Let $x(0) = \frac{s_0}{\delta+n}$. Find the particular solution $(x(t))$ to this ODE.

$$x(0) = a + \frac{s_1}{(\delta+n)}$$

$$a = x(0) - \frac{s_1}{(\delta+n)}$$

$$a = \frac{s_0 - s_1}{(\delta+n)}$$

Thus, the particular solution is:

$$x(t) = \frac{s_0 - s_1}{(\delta+n)}\ e^{-(\delta+n)(1-\alpha)t} + \frac{s_1}{(\delta+n)}$$